PERFORMANCE EVALUATION OF A STOCHASTIC SUPPLY CHAIN

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ABSTRACT
This paper concentrates on a multicommodity stochastic-flow network in which each arc has both the capacity and cost attributes. We utilize this network to model the supply chain with several factories. Different from the single-commodity case, the system capacity is a pattern for multicommodity case. We propose a new performance index, the probability that the system capacity is less than or equal to a given pattern under the budget constraint. A simple algorithm based on minimal cuts is presented to generate all (**d**,*B*)-MCs that are the maximal capacity vectors meeting the demand **d** and budget *B*. The proposed performance index, which is evaluated in terms of (**d**,*B*)-MCs, is applied to evaluate the performance of a supply chain.

KEY WORDS
Performance, Supply Chain, Minimal Cuts, Stochastic-Low Network, Multicommodity

1. Introduction

Traditionally, the system capacity in single-commodity case is the maximum flow from the source *s* to the sink *t*. In many real networks, the arc capacity should be stochastic due to the fact that the arc may be in failure, maintenance, etc. Such a network is called a single-commodity stochastic-flow network. Hence, its system capacity is not a fixed number. Without any budget constraint, several authors [11,13,20,21,23] presented algorithms to generate all *d*-MPs (or named lower boundary points for *d*) for the demand *d* in order to evaluate a performance index, the probability that the system capacity is larger than or equal to *d*.

A MC is a set of arcs whose proper subsets are no longer cuts. Jane *et al.* [9], Lin [14] and Yeh [22] used MCs to generate all *d*-MCs (or named upper boundary points for *d*) in order to evaluate another performance index, the probability that the system capacity is less than or equal to *d*. Moreover, many flow networks allow multiple types of commodity to be transmitted from *s* to *t* simultaneously. Such a network is called a multicommodity stochastic-flow network (MSFN) throughout this paper. Under the assumption that the capacity of each arc is deterministic, Assad [2], Ford and Fulkerson [5], Held *et al.* [7], Hu [8], Jarvis [10] and Rothechild and Whinston [18] solved the multicommodity maximum flow (MMF) problem to find the maximal total flow with arc-capacity constraints. However, the maximal total flow is not a suitable performance index if different type of commodity consumes the capacity of each arc differently.

This paper concentrates on the performance evaluation for a MSFN with the budget constraint. Each arc has both the capacity and cost attributes. Such network is constructed to model the supply chain with several factories. We firstly define the system capacity as a vector and then propose a new performance index, the probability that the system capacity is less than or equal to a given pattern **d** under the budget *B*. In order to evaluate such a performance index, an algorithm in terms of MCs is presented to generate all (**d**,*B*)-MCs, which are the maximal capacity vectors meeting the demand **d** and the budget *B*.

2. Supply Chain Model

Let $G(A, M, C)$ denote a MSFN where $A \equiv \{a_i | 1 \le i \le n\}$ is the set of arcs, $M \equiv (M_1, M_2, \ldots, M_n)$ with M_i the maximal capacity of a_i, and $C \equiv (c_1, c_2, \ldots, c_n)$ with c_i the cost per unit of capacity on a_i. The notation x_i denotes the (current) capacity of arc a_i, and $X \equiv (x_1, x_2, \ldots, x_n)$ denotes the capacity vector of G. Let α_k denote the consumed quantity of capacity on each arc by per commodity k, $k = 1, 2, \ldots, p$. Without loss of generality for α_k, we assume that $\alpha_p \ge \alpha_{p-1} \ge \ldots \ge \alpha_1 = 1$. G is required to further satisfy the following assumptions:

1. The flows in G must satisfy the flow-conservation law [4].
2. The capacities of different arcs are statistically independent.

3. p types of commodity are transmitted from s to t.
4. Each node is perfectly reliable.

Vector comparisons will be done as follows throughout this paper.

$Y \geq X$ $(y_1, y_2, \ldots, y_n) \geq (x_1, x_2, \ldots, x_n)$: $y_i \geq x_i$ for each $i = 1, 2, \ldots, n$

$Y > X$ $(y_1, y_2, \ldots, y_n) > (x_1, x_2, \ldots, x_n)$: $Y \geq X$ and $y_i > x_i$ for at least one

Suppose that $K_1, K_2, \ldots, K_m$ are m MCs of G. With respect to each MC $K_r = \{a_{r1}, a_{r2}, \ldots, a_{rn_r}\}$ where n_r is the number of arcs in K_r, the vector $F_r = (F_r^1, F_r^2, \ldots, F_r^p)$ is called a flow assignment where $F_r^k = (f_{r1}^k, f_{r2}^k, \ldots, f_{rn_r}^k)$ with f_{rj}^k denoting the flow of commodity k through a_{rj}, $j = 1, 2, \ldots, n_r$, $k = 1, 2, \ldots, p$. It is feasible under the capacity vector $M = (M_1, M_2, \ldots, M_n)$ if

$$\sum_{k=1}^{p} \alpha_k f_{rj}^k \leq M_{rj} \quad \text{for } j = 1, 2, \ldots, n_r. \qquad (1)$$

Similarly, any flow assignment F_r is feasible under the capacity vector $X = (x_1, x_2, \ldots, x_n)$ if

$$\sum_{k=1}^{p} \alpha_k f_{rj}^k \leq x_{rj} \quad \text{for } j = 1, 2, \ldots, n_r. \qquad (2)$$

Under the capacity vector X, the MC K_r is said to support the demand $\mathbf{d} = (d_1, d_2, \ldots, d_p)$ if there exists an F_r feasible under X such that

$$\sum_{j=1}^{n_r} f_{rj}^k = d_k \quad \text{for } k = 1, 2, \ldots, p, \qquad (3)$$

where d_k is the required quantity of commodity k at t, $k = 1, 2, \ldots, p$. Under X, K_r is said to support at most $\mathbf{d}$ if K_r supports $\mathbf{d}$ and K_r cannot support $(\mathbf{d} + e_1)$ where e_i is a p-tuple vector that has 1 at position i and 0 at all other positions. The capacity vector X is said to support $\mathbf{d}$ if under X, all MCs support $\mathbf{d}$. Under X, if all MCs support $\mathbf{d}$ and at least one MC supports at most $\mathbf{d}$, then X supports $\mathbf{d}$ but cannot support any $\mathbf{d}'$ with $\mathbf{d}' > \mathbf{d}$. That is, X supports at most $\mathbf{d}$.

For a MSFN, the system capacity $V(X)$ is defined to be $\mathbf{d}$ if X supports at most $\mathbf{d}$. Two performance indexes: $\Pr\{V(X) \geq \mathbf{d}\}$ and $\Pr\{V(X) \leq \mathbf{d}\}$ can be adopted to evaluate the quality level of a MSFN. Lin [15] proposed an algorithm to evaluate $\Pr\{V(X) \geq \mathbf{d}\}$ without any budget constraint in terms of MPs. We try to evaluate $\Pr\{V(X) \leq \mathbf{d}\}$ subject to budget B in terms of MCs. Let $C(X) \equiv \sum_{i=1}^{n} c_i x_i$ the total cost under the capacity X.

We define the capacity vector X as a $(\mathbf{d},B)$-MC if i) $V(X) = \mathbf{d}$, ii) $C(X) \leq B$ and iii) $V(Y) > \mathbf{d}$ or $C(Y) > B$ for any capacity vector Y with $Y > X$. With respect to each K_r, we generate an X via (5) for each flow assignment F_r satisfying (4). Such an X supporting at most $\mathbf{d}$ and meeting $C(X) \leq B$ is a candidate of $(\mathbf{d},B)$-MC. Let Ω be the set of such candidates. Apparently, $\Omega \subset \{X | V(X) = \mathbf{d} \;\&\; C(X) \leq B\}$.

3. Algorithm

As those approaches proposed by Jane *et al.* [9], Lin [14], Xue [20] and Yeh [22], we suppose all MCs have been determined in advance.

Step1. With respect to each MC $K_r = \{a_{r1}, a_{r2}, \ldots, a_{rn_r}\}$.

1.1) Find all flow assignments $F_r = (F_r^1, F_r^2, \ldots, F_r^p)$ with $F_r^k = (f_{r1}^k, f_{r2}^k, \ldots, f_{rn_r}^k)$, $k = 1, 2, \ldots, p$, which satisfy the arc-capacity and demand constraints,

$$\sum_{k=1}^{p} \alpha_k f_{rj}^k \leq M_{rj} \quad \text{for } j = 1, 2, \ldots, n_r, \qquad (4)$$

$$\sum_{j=1}^{n_r} f_{rj}^k = d_k \qquad k = 1, 2, \ldots, p. \qquad (5)$$

1.2) (Obtain Ω) Transform each flow assignment F_r into $X = (x_1, x_2, \ldots, x_n)$ via

$$\begin{cases} x_{rj} = \sum_{k=1}^{p} \alpha_k f_{rj}^k & \text{for } j = 1, 2, \ldots, n_r, \\ x_i = M_i & \text{for } a_i \notin K_r. \end{cases} \qquad (6)$$

and store X into Ω if $\sum_{i=1}^{n} c_i x_i \leq B$.

Step2. (Obtain $\Omega_{\max}$) Suppose $\Omega = \{X_1, X_2, \ldots, X_z\}$. Remove those non-maximal ones from Ω to obtain $\Omega_{\max}$.

2.1) $\mathrm{I} \leftarrow \phi$

2.2) For $i \leftarrow 1$ to z with $i \notin \mathrm{I}$

2.3) For $j \leftarrow i + 1$ to z with $j \notin \mathrm{I}$

2.4) If $X_j \geq X_i$, then $X_i \notin \Omega_{\max}$, $\mathrm{I} \leftarrow \mathrm{I} \cup \{i\}$ and Goto 2.7)

ElseIf $X_i > X_j$, then $\mathrm{I} \leftarrow \mathrm{I} \cup \{j\}$.

2.5) $j \leftarrow j + 1$
2.6) X_i is a (**d**,B)-MC.
2.7) $i \leftarrow i + 1$
2.8) END.

If $X_1, X_2, ..., X_v$ are (**d**,B)-MCs, let $B_i = \{X | X \le X_i\}$ for $i = 1, 2, \ldots, v$. Thus, the performance index $U_{\mathbf{d},B} = \Pr\{B_1 \cup B_2 \cup \ldots \cup B_v\}$. It can be calculated by applying methods such as inclusion-exclusion method [6,13-16], disjoint subsets [13,14,20] and state-space decomposition [9,11,12].

4. A Supply Chain Example

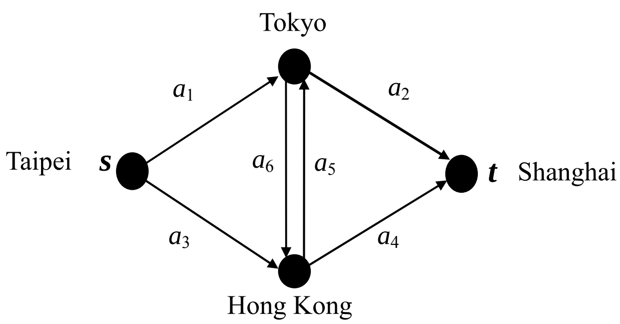

Figure 1: A Trade Network

Figure 1 shows a supply chain network. A manufacturing firm would like to transmit 15-inch LCD monitors (commodity 1) and 17-inch LCD monitors (commodity 2) from Taipei to Shanghai passing Hong Kong or Tokyo. One unit of commodity means 120 commodities of the same type. The firm wants to have its 2 units of commodity 1 and 2 units of commodity 2 (i.e., $d_1 = 2$ & $d_2 = 2$) to be transported. One unit of commodity 1 (resp. 2) consumes 1 (resp. 2) container, i.e., $\alpha_1 = 1$ (resp. $\alpha_2 = 2$). The data for arcs are shown in Table 1. In this example, there are four MCs: $K_1 = \{a_1, a_3\}$, $K_2 = \{a_2, a_4\}$, $K_3 = \{a_1, a_4, a_5\}$ & $K_4 = \{a_2, a_3, a_6\}$. The agent manager would like to evaluate the performance index $U_{\mathbf{d},B}$ if the budget is 2550 US dollars.

Table 1: The Data of Arcs for Figure 1

Arc	Capacity	Cumulative probability	Cost**	α_1	α_2
a_1	4	1.00	150		
	3*	0.50			
	2	0.30			
	1	0.20			
	0	0.10			
a_2	3	1.00	150		
	2	0.40		1	2
	1	0.20			
	0	0.10			
a_3	3	1.00	180		
	2	0.40			
	1	0.20			
	0	0.10			
a_4	4	1.00	90		
	3	0.50			
	2	0.30			
	1	0.20			
	0	0.10			
a_5	3	1.00	150		
	2	0.40		1	2
	1	0.20			
	0	0.10			
a_6	3	1.00	90		
	2	0.40			
	1	0.20			
	0	0.10			

*$\Pr\{x_1 \le 3\} = 0.5$ ** US dollars

Step 1. With respect to $K_1 = \{a_1, a_3\}$.

1.1) Find all flow assignments $F_1 = (f_1^1, f_3^1, f_1^2, f_3^2)$ satisfying the following constraints.

$$\begin{cases} f_1^1 + 2f_1^2 \le 4 \\ f_3^1 + 2f_3^2 \le 3 \end{cases}$$

$$\begin{cases} f_1^1 + f_3^1 = 2 \\ f_1^2 + f_3^2 = 2 \end{cases}$$

We obtain three $(f_1^1, f_3^1, f_1^2, f_3^2)$: (1,1,1,1), (2,0,1,1) & (0,2,2,0).

1.2) Transform each $(f_1^1, f_3^1, f_1^2, f_3^2)$ into $X = (x_1, x_2, \ldots, x_6)$ via

$$\begin{cases} x_1 = f_1^1 + 2f_1^2 \ , \ x_3 = f_3^1 + 2f_3^2, \\ x_2 = M_2 \ , \ x_4 = M_4, x_5 = M_5, x_6 = M_6. \end{cases}$$

Two different X are generated: $X_1 = (3,3,3,4,3,3)$ & $X_2 = (4,2,3,4,3,3)$. Both $C(X_1) = 2520$ and $C(X_2) = 2490$ do not exceed the budget, so $\Omega = \{X_1, X_2\}$.

- With respect to $K_2 = \{a_2, a_4\}$.

1.1) Obtain all $F_2 = (f_2^1, f_4^1, f_2^2, f_4^2)$ of the following constraints.

$$\begin{cases} f_2^1 + 2f_2^2 \le 3 \\ f_4^1 + 2f_4^2 \le 4 \end{cases}$$

$$\begin{cases} f_2^1 + f_4^1 = 2 \\ f_2^2 + f_4^2 = 2 \end{cases}$$

We obtain three $(f_2^1, f_4^1, f_2^2, f_4^2)$: (1,1,1,1), (2,0,0,2) & (0,2,1,1).

1.2) Transform each ($f_2^1, f_4^1, f_2^2, f_4^2$) into $X = (x_1, x_2, \ldots, x_6)$ via $x_2 = f_2^1 + 2f_2^2$, $x_4 = f_4^1 + 2f_4^2$ and $x_i = M_i$ for others. Then two different X are generated: (4,3,3,3,3,3) & (4,3,2,4,3,3). But the total cost of (4,3,3,3,3,3) is 2580. So X_3 = (4,3,2,4,3,3) and $\Omega = \{X_1, X_2, X_3\}$.

- With respect to $K_3 = \{a_1, a_4, a_5\}$.

In sum we obtain 28 candidates. The results are shown in Table 2.

Step 2. $\Omega = \{X_1, X_2, \ldots, X_{28}\}$.

2.1) $I \leftarrow \phi$

2.2) $i \leftarrow 1$

2.3) $j \leftarrow 2$

2.4) $X_2 = (4,2,3,4,3,3) \ngeq X_1 = (3,3,3,4,3,3)$ and $X_1 \ngtr X_2$. $I \leftarrow \phi$.

2.3) $j \leftarrow 3$

2.4) $X_3 \ngeq X_1$ and $X_1 \ngtr X_3$. $I \leftarrow \phi$.

2.8) END.

Table 2: The Fesults for Generating (2,2,2550)-MCs

MC	Candidate of (2,2,2550)-MC	$C(X)$	Is a (2,2,2550)-MC
K_1	X_1 = (3,3,3,4,3,3)	2520	*YES*
	X_2 = (4,3,2,4,3,3)	2490	*YES*
K_2	X_3 = (4,2,3,4,3,3)	2520	*YES*
K_3	X_4 = (2,3,3,4,0,3)	1920	NO
	X_5 = (2,3,3,2,2,3)	2040	NO
	X_6 = (4,3,3,2,0,3)	2040	*YES*
	X_7 = (4,3,3,0,2,3)	2160	*YES*
	X_8 = (0,3,3,4,2,3)	1920	NO
	X_9 = (3,3,3,3,0,3)	2160	NO
	X_{10} = (3,3,3,1,2,3)	2100	NO
	X_{11} = (1,3,3,3,2,3)	1980	NO
	X_{12} = (1,3,3,4,1,3)	1920	NO
	X_{13} = (3,3,3,2,1,3)	2040	NO
	X_{14} = (3,3,3,0,3,3)	2160	NO
	X_{15} = (0,3,3,3,3,3)	1980	NO
	X_{16} = (1,3,3,2,3,3)	2040	NO
	X_{17} = (2,3,3,1,3,3)	2100	NO
	X_{18} = (2,3,3,3,1,3)	1980	NO
K_4	X_{19} = (4,2,2,4,3,2)	2250	NO
	X_{20} = (4,3,3,4,3,0)	2400	*YES*
	X_{21} = (4,3,1,4,3,2)	2220	NO
	X_{22} = (4,1,3,4,3,2)	2280	NO
	X_{23} = (4,3,2,4,3,1)	2310	NO
	X_{24} = (4,3,0,4,3,3)	2130	NO
	X_{25} = (4,0,3,4,3,3)	2220	NO
	X_{26} = (4,1,2,4,3,3)	2190	NO
	X_{27} = (4,2,1,4,3,3)	2160	NO
	X_{28} = (4,2,3,4,3,1)	2340	NO

Six candidates are (2,2,2550)-MCs: X_1, X_2, X_3 X_6, X_7 and X_{20}. In order to compute $U_{2,2,2550} = \Pr\{X|V(X) \le (2,2)$ & $C(X) \le 2550\}$, we let $B_1 = \{X|X \le X_1\}$, $B_2 = \{X|X \le X_2\}$, $B_3 = \{X|X \le X_3\}$, $B_4 = \{X|X \le X_6\}$, $B_5 = \{X|X \le X_7\}$ & $B_6 = \{X|X \le X_{20}\}$. Then $U_{2,2,2550} = \Pr\{B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6\} = 0.84772$ after calculation.

5. Summary and Discussion

This paper applies the properties of MCs to define the system capacity $V(X)$ as $(d_1, d_2, \ldots, d_p)$ if at most d_k quantity of commodity k can be transmitted simultaneously under X, $k = 1, 2, \ldots, p$. A simple algorithm is proposed to generate all ($\mathbf{d}$,B)-MCs. The performance index $U_{\mathbf{d},B}$, the probability that $V(X)$ is less than or equal to $\mathbf{d}$ subject to the budget B, can then be computed in terms of ($\mathbf{d}$,B)-MCs.

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